Journal of Difference Equations and Applications, 2013
http://dx.doi.org/10.1080/10236198.2013.766729

Hille and Nehari type criteria for third-order delay dynamic equations†

Ravi P. Agarwal[a1], Martin Bohner[b]*, Tongxing Li[c2] and Chenghui Zhang[c3]

[a]*Department of Mathematics, Texas A&M University-Kingsville, University Blvd., Kingsville, TX 78363-8202, USA;* [b]*Department of Mathematics and Statistics, Missouri S&T, Rolla, MO 65409-0020, USA;* [c]*School of Control Science and Engineering, Shandong University, Jinan, Shandong 250061, P.R. China*

(*Received 30 September 2012; final version received 4 January 2013*)

The objective of this note is to present new Hille and Nehari type asymptotic criteria for a class of third-order delay dynamic equations on a time scale. Assumptions in our theorems are less restrictive, whereas the proofs are significantly simpler compared to those reported in the literature. The results obtained extend and improve some previous results.

Keywords: asymptotic behaviour; delay dynamic equation; third-order equation; time scale

AMS Subject Classification: 34K11; 34N05; 39A10

1. Introduction

The theory of dynamic equations on time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph.D. thesis [22] in order to unify continuous and discrete analysis. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them is the study of population dynamic models (see [6]). Not only this theory of the so-called dynamic equations can unify the theories of differential equations and difference equations, but also it is able to extend these classical cases to cases 'in between', e.g., to the so-called q-difference equations. Several authors have expounded on various aspects of this new theory (see the survey paper by Agarwal et al. [1] and references cited therein). The books on the subject of time scales, by Bohner and Peterson [6,7], summarize and organize much of the time scale calculus and some applications in the real world.

For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in asymptotic behaviour, we suppose that the time scale under consideration is not bounded above and is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. On any time scale, we define the forward and backward jump operators by $\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} | s < t\}$, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, $\emptyset$ denotes the

†This paper is dedicated to Professor Gerasimos Ladas on the occasion of his retirement.

*Corresponding author. Email: bohner@mst.edu

empty set. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$, we denote $f^{\sigma}(t) := f(\sigma(t))$. For some other concepts related to the notion of time scales, see [1,6,7,22].

Recently, there has been an increasing interest in obtaining sufficient conditions for oscillatory or nonoscillatory behaviour of solutions of different classes of second-order dynamic equations [2,4,5,8,16,17,18,24,27–30] and third-order dynamic equations [3,9,11,12,13–15,19–21,23,25,26,31–35] on time scales. In the following, we introduce the background details that serve the contents of this paper. Regarding oscillation and asymptotic behaviour of third-order dynamic equations, Erbe et al. [13] considered a third-order ordinary dynamic equation

$$(a(rx^{\Delta})^{\Delta})^{\Delta}(t) + p(t)f(x(t)) = 0, \tag{1.1}$$

where $f \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ is assumed to satisfy $uf(u) > 0$ and $f(u)/u \geq K > 0$ for $u \neq 0$, a, r and p are positive real-valued rd-continuous functions which satisfy

$$\int_{t_0}^{\infty} \frac{\Delta t}{a(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty. \tag{1.2}$$

They proved several oscillation criteria for (1.1), one of which we present below for the convenience of the reader.

Theorem 1.1. (*See* [13, *Theorem* 1 *and Remark* 1]) *Let* (1.2) *hold and* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *be large enough. Assume that there exists a positive differentiable function* η *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(K\eta(s)p(s) - \frac{r(s)(\eta^{\Delta}(s))^2}{4\eta(s)\int_{t_1}^{s}(\Delta u/a(u))} \right) \Delta s = \infty.$$

Then the solution x *of* (1.1) *is oscillatory or* $\lim_{t\to\infty}x(t)$ *exists* (*finite*).

Yu and Wang [34] extended results of [13] to a general third-order dynamic equation

$$(a((r(x^{\Delta})^{\gamma_1})^{\Delta})^{\gamma_2})^{\Delta}(t) + f(x(t)) = 0.$$

In 2007, Erbe et al. [15] investigated a third-order dynamic equation

$$x^{\Delta\Delta\Delta}(t) + p(t)x(t) = 0, \tag{1.3}$$

where p is a positive real-valued rd-continuous function defined on $\mathbb{T}$, and established some Hille and Nehari type oscillation criteria for (1.3), one of which we give below.

Theorem 1.2. (*See* [15, *Theorem* 2]) *Assume that*

$$\int_{t_0}^{\infty}\int_{z}^{\infty}\int_{u}^{\infty} p(s)\Delta s\Delta u\Delta z = \infty \tag{1.4}$$

and

$$\liminf_{t\to\infty} t\int_t^\infty \frac{h_2(s,t_0)}{\sigma(s)}p(s)\Delta s > \frac{1}{4}, \tag{1.5}$$

where $h_2(t,s)$ *is the Taylor monomial of degree* 2 (*see Bohner and Peterson* [6, *Section* 1.6]). *Then the solution* x *of* (1.3) *is oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$.

Very recently, Wang and Xu [33] have considered a generalized ordinary dynamic equation

$$(r_2((r_1x^\Delta)^\Delta)^\gamma)^\Delta(t) + q(t)f(x(t)) = 0, \tag{1.6}$$

where $\gamma \geq 1$ is the ratio of odd positive integers; r_1, r_2 and q are positive rd-continuous functions defined on $\mathbb{T}$, $f \in \mathrm{C}(\mathbb{R},\mathbb{R})$, and there exists a positive number M such that $f(u)/u^\gamma \geq M > 0$ for $u \neq 0$. They obtained some new Hille and Nehari type results provided that

$$\lim_{t\to\infty}\frac{P^\sigma(t)}{P(t)} = 1, \tag{1.7}$$

where

$$\delta_1(t,t_1) := \int_{t_1}^t \frac{\Delta s}{r_2^{1/\gamma}(s)}, \qquad \delta_2(t,t_1) := \int_{t_1}^t \frac{\delta_1(s,t_1)}{r_1(s)}\Delta s,$$

$$p(t) := \frac{\gamma}{r_1(t)}\delta_1(t,t_1)\delta_2^{\gamma-1}(t,t_1), \quad \text{and} \quad P(t) := \int_{t_1}^t p(s)\Delta s.$$

Note that (1.7) depends on a concrete time scale, i.e., $\sigma(t)$. For oscillation of third-order dynamic equations with deviating arguments, Elabbasy and Hassan [9] investigated a third-order dynamic equation

$$(ax^{\Delta^2})^\Delta(t) + p(t)x(\tau(t)) = 0$$

in the case where $\int_{t_0}^\infty p(t)\tau(t)\Delta t = \infty$. Han et al. [20] considered a third-order dynamic equation

$$((x^{\Delta^2})^\gamma)^\Delta(t) + p(t)x^\gamma(\tau(t)) = 0.$$

Li et al. [26] investigated a third-order dynamic equation

$$(r(x^{\Delta^2})^\gamma)^\Delta(t) + p(t)x^\gamma(\tau(t)) = 0$$

under the case where

$$\int_{t_0}^\infty p(t)\tau^\gamma(t)\Delta t = \infty.$$

Agarwal et al. [3], Erbe et al. [11,12], Hassan [21], Kubiaczyk and Saker [23], Li et al. [25] and Saker [31,32] examined a third-order dynamic equation

$$(a((rx^{\Delta})^{\Delta})^{\gamma})^{\Delta}(t)+f(t,x(\tau(t)))=0, \tag{1.8}$$

where $\gamma>0$ is the ratio of odd positive integers, a and r are positive rd-continuous functions defined on $\mathbb{T}$, $\tau\in C_{rd}(\mathbb{T},\mathbb{T})$, $\tau(t)\le t$, $\lim_{t\to\infty}\tau(t)=\infty$, $f\in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ is assumed to satisfy $uf(t,u)>0$ for $u\neq 0$, and there exists a positive rd-continuous function p defined on $\mathbb{T}$ such that $f(t,u)/u^{\gamma}\ge p(t)$ for $u\neq 0$. In [12,21], the authors established some oscillation criteria for (1.8) in the case where

$$\tau(\sigma(t))=\sigma(\tau(t)) \quad \text{and} \quad \tau^{\Delta}(t)>0. \tag{1.9}$$

For the convenience of the reader, we introduce a result in [21].

THEOREM 1.3. (*See* [21, *Corollary* 2.3]) *Let* $\gamma\ge 1$ *and* (1.9) *hold. Assume that*

$$\int_{t_0}^{\infty}\frac{\Delta t}{a^{1/\gamma}(t)}=\infty, \quad \int_{t_0}^{\infty}\frac{\Delta t}{r(t)}=\infty,$$

and

$$\int_{t_0}^{\infty}\frac{1}{r(t)}\int_{t}^{\infty}\left[\frac{1}{a(s)}\int_{s}^{\infty}p(u)\Delta u\right]^{1/\gamma}\Delta s\Delta t=\infty.$$

If there exists a positive differentiable function η *such that*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\eta(s)p(s)-\frac{r^{\gamma}(\tau(s))(\eta^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\eta^{\gamma}(s)(\tau^{\Delta}(s))^{\gamma}\left(\int_{t_1}^{\tau(s)}(\Delta u/(a^{1/\gamma}(u)))\right)^{\gamma}}\right)\Delta s=\infty$$

for $t_2>t_1>t_0$, *then the solution* x *of* (1.8) *is oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$.

Note that $\tau(\sigma(t))=\sigma(\tau(t))$ depends on time scales and can be a restriction for applications. In order to expurgate this assumption, Li et al. [25] obtained some oscillation results for (1.8) provided that

$$r^{\Delta}(t)\le 0 \quad \text{and} \quad \int_{t_0}^{\infty}p(t)\tau^{\gamma}(t)\Delta t=\infty.$$

Saker [31,32] established some new criteria for oscillation of (1.8) in the case where $a^{\Delta}(t)\ge 0$ and $r(t)=1$. In particular, Saker [31] presented some Hille and Nehari type oscillation results for (1.8), some of which we present below for the convenience of the reader. In what follows, we use the notation

$$A(t):=p(t)\left(\frac{h_2(\tau(t),t_0)}{\sigma(t)}\right)^{\gamma}, \quad A_*:=\liminf_{t\to\infty}\frac{t^{\gamma}}{a(t)}\int_{\sigma(t)}^{\infty}A(s)\Delta s,$$

$$B_*:=\liminf_{t\to\infty}\frac{1}{t}\int_{T}^{t}\frac{s^{\gamma+1}}{a(s)}A(s)\Delta s \quad \text{and} \quad l:=\liminf_{t\to\infty}\frac{t}{\sigma(t)}$$

for $T\ge t_0$.

THEOREM 1.4. (*See* [31, *Theorem* 3.4]) *Let* $a^{\Delta}(t) \geq 0$, $r(t) = 1$,

$$\int_{t_0}^{\infty} \frac{\Delta t}{a^{1/\gamma}(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty}\int_{z}^{\infty}\left[\frac{1}{a(u)}\int_{u}^{\infty} p(s)\Delta s\right]^{1/\gamma}\Delta u \Delta z = \infty.$$

Assume that

$$A_* > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}},$$

or

$$A_* + B_* > \frac{1}{l^{\gamma^2+\gamma}}.$$

Then the solution x *of* (1.8) *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

THEOREM 1.5. (*See* [31, *Corollary* 3.5]) *Let* (1.4) *hold,* $a(t) = r(t) = 1$, $\gamma = 1$ *and* $f(t,u) = u$. *If*

$$\liminf_{t\to\infty} t \int_{\sigma(t)}^{\infty} p(s)\frac{h_2(\tau(s), t_0)}{\sigma(s)}\Delta s > \frac{1}{4l}, \tag{1.10}$$

then the solution x *of* (1.8) *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

As a special case when $\tau(t) = t$, condition (1.10) reduces to

$$\liminf_{t\to\infty} t \int_{\sigma(t)}^{\infty} p(s)\frac{h_2(s, t_0)}{\sigma(s)}\Delta s > \frac{1}{4l}. \tag{1.11}$$

Comparing condition (1.5) with condition (1.11) reveals that the result in [15] improves that of [31] in the sense that

$$\frac{1}{4} \leq \frac{1}{4l} \quad \text{and} \quad t\int_{t}^{\infty} \frac{h_2(s,t_0)}{\sigma(s)}p(s)\Delta s \geq t\int_{\sigma(t)}^{\infty} p(s)\frac{h_2(s,t_0)}{\sigma(s)}\Delta s.$$

So to sum up the above details, Erbe et al. [15] gave the best Hille and Nehari type oscillation criteria for third-order dynamic equations.

As demonstrated in [33], results obtained in [15] require auxiliary functions, e.g., $h_2(t, t_0)$. The natural question now is: Can one find other new Hille and Nehari type criteria without (1.7), some restrictive assumptions on coefficients and any auxiliary functions for a generalized third-order delay dynamic equation? Our objective of this paper is to give an affirmative answer to this question and consider a third-order delay dynamic equation

$$(a(rx^{\Delta})^{\Delta})^{\Delta}(t) + p(t)x(\tau(t)) = 0 \tag{1.12}$$

on an arbitrary time scale $\mathbb{T}$. Throughout, we always assume that a, r and p are positive real-valued rd-continuous functions defined on $\mathbb{T}$, $\tau \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$, $\tau(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \infty$.

By a solution of (1.12) we mean a nontrivial, realvalued function $x \in \mathrm{C}^1_{\mathrm{rd}}([t_x,\infty)_{\mathbb{T}})$, $t_x \in [t_0,\infty)_{\mathbb{T}}$ which satisfies (1.12). The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution x of (1.12) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

Below, all occurring functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

2. Main results

In this section, we establish some new asymptotic criteria for (1.12). Before stating the main results, we begin with the following lemma which is important in the proofs of the main results.

Lemma 2.1. *Assume that*

$$\int_{t_0}^{\infty} \frac{\Delta t}{a(t)} = \int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty. \tag{2.1}$$

Suppose also that x is an eventually positive solution of (1.12). *Then there are only the following two cases for $t \in [t_1, \infty)_{\mathbb{T}} \subseteq [t_0, \infty)_{\mathbb{T}}$ sufficiently large:*

(1) $x(t) > 0$, $x^{\Delta}(t) > 0$, $(rx^{\Delta})^{\Delta}(t) > 0$, $(a(rx^{\Delta})^{\Delta})^{\Delta}(t) < 0$;
(2) $x(t) > 0$, $x^{\Delta}(t) < 0$, $(rx^{\Delta})^{\Delta}(t) > 0$, $(a(rx^{\Delta})^{\Delta})^{\Delta}(t) < 0$.

Proof. The proof is similar to that of [13, Lemma 1], and hence is omitted. □

The next lemma can be considered as a generalization of [15, Lemma 4].

Lemma 2.2. *Assume that x satisfies case* (1) *of Lemma* 2.1. *Then*

$$x(t) \geq \left(\frac{r(t)}{\int_{t_1}^{t} (\Delta s/a(s))} \int_{t_2}^{t} \frac{\int_{t_1}^{s} (\Delta u/a(u))}{r(s)} \Delta s \right) x^{\Delta}(t)$$

for $t \in [t_2, \infty)_{\mathbb{T}} \subseteq (t_1, \infty)_{\mathbb{T}}$ sufficiently large and $r(t)x^{\Delta}(t)/\int_{t_1}^{t} (\Delta s/a(s))$ is nonincreasing eventually.

Proof. Assume that x satisfies case (1) of Lemma 2.1. Then we get

$$r(t)x^{\Delta}(t) = r(t_1)x^{\Delta}(t_1) + \int_{t_1}^{t} \frac{a(s)(rx^{\Delta})^{\Delta}(s)}{a(s)} \Delta s \geq \left(a(t) \int_{t_1}^{t} \frac{\Delta s}{a(s)} \right) (rx^{\Delta})^{\Delta}(t)$$

for $t \in [t_2, \infty)_{\mathbb{T}} \subseteq [t_1, \infty)_{\mathbb{T}}$ sufficiently large. Hence we have

$$\left(\frac{r(t)x^{\Delta}(t)}{\int_{t_1}^{t} (\Delta s/a(s))} \right)^{\Delta} \leq 0,$$

which implies that

$$x(t)=x(t_2)+\int_{t_2}^{t}\frac{r(s)x^{\Delta}(s)}{\int_{t_1}^{s}(\Delta u/a(u))}\frac{\int_{t_1}^{s}(\Delta u/a(u))}{r(s)}\Delta s\geq\left(\frac{r(t)}{\int_{t_1}^{t}(\Delta s/a(s))}\int_{t_2}^{t}\frac{\int_{t_1}^{s}(\Delta u/a(u))}{r(s)}\Delta s\right)x^{\Delta}(t).$$

This completes the proof. □

Lemma 2.3. *Assume that x is a solution of* (1.12) *which satisfies case* (2) *of Lemma* 2.1. *If*

$$\int_{t_0}^{\infty}\frac{1}{r(z)}\int_{z}^{\infty}\frac{1}{a(u)}\int_{u}^{\infty}p(s)\Delta s\Delta u\Delta z=\infty, \tag{2.2}$$

then $\lim_{t\to\infty}x(t)=0$.

Proof. The proof is similar to that of [3, Lemma 2.1], and so is omitted. □

Remark 2.4. In fact, assume that x is a solution of (1.12) which satisfies case (2) of Lemma 2.1. If

$$\int_{t_0}^{\infty}p(t)\Delta t=\infty$$

or

$$\int_{t_0}^{\infty}p(t)\Delta t<\infty \quad \text{and} \quad \int_{t_0}^{\infty}\frac{1}{a(u)}\int_{u}^{\infty}p(s)\Delta s\Delta u=\infty$$

holds, then $\lim_{t\to\infty}x(t)=0$ also holds.

Lemma 2.5. *Assume that x is a solution of* (1.12) *which satisfies case* (1) *of Lemma* 2.1. *Define the Riccati transformation*

$$\omega(t):=\frac{a(t)(rx^{\Delta})^{\Delta}(t)}{r(t)x^{\Delta}(t)}. \tag{2.3}$$

Then there exist t_1, t_2 and $t_3\in[t_0,\infty)_{\mathbb{T}}$ such that $[t_3,\infty)_{\mathbb{T}}\subseteq[t_2,\infty)_{\mathbb{T}}\subseteq[t_1,\infty)_{\mathbb{T}}$, $\tau(t)\in(t_2,\infty)_{\mathbb{T}}$ for $t\in[t_3,\infty)_{\mathbb{T}}$, and

$$\omega^{\Delta}(t)+\frac{\int_{t_2}^{\tau(t)}\frac{\int_{t_1}^{s}(\Delta u/a(u))}{r(s)}\Delta s}{\int_{t_1}^{\sigma(t)}(\Delta s/a(s))}p(t)+\frac{\omega^2(t)}{a(t)}\frac{\int_{t_1}^{t}(\Delta s/a(s))}{\int_{t_1}^{\sigma(t)}(\Delta s/a(s))}\leq 0, \tag{2.4}$$

$$\omega^{\Delta}(t)+\frac{\int_{t_2}^{\tau(t)}\frac{\int_{t_1}^{s}(\Delta u/a(u))}{r(s)}\Delta s}{\int_{t_1}^{\sigma(t)}(\Delta s/a(s))}p(t)+\frac{\omega^{\sigma}(t)\omega(t)}{a(t)}\leq 0, \tag{2.5}$$

$$\omega^{\Delta}(t)+\frac{\int_{t_2}^{\tau(t)}\frac{\int_{t_1}^{s}(\Delta u/a(u))}{r(s)}\Delta s}{\int_{t_1}^{\sigma(t)}(\Delta s/a(s))}p(t)+\frac{\omega^2(t)}{a(t)+\mu(t)\omega(t)}\leq 0 \tag{2.6}$$

for $t\in[t_3,\infty)_{\mathbb{T}}$.

Proof. Let x be as in the statement of this lemma. Then we have

$$\begin{aligned}\omega^{\Delta}(t) &= \left(\frac{a(rx^{\Delta})^{\Delta}}{rx^{\Delta}}\right)^{\Delta}(t) = \frac{(a(rx^{\Delta})^{\Delta})^{\Delta}(t)r(t)x^{\Delta}(t) - a(t)(rx^{\Delta})^{\Delta}(t)(rx^{\Delta})^{\Delta}(t)}{r(t)x^{\Delta}(t)(rx^{\Delta})^{\sigma}(t)} \\ &= -\frac{x(\tau(t))}{(rx^{\Delta})^{\sigma}(t)}p(t) - \frac{(rx^{\Delta})^{\Delta}(t)}{(rx^{\Delta})^{\sigma}(t)}\omega(t).\end{aligned} \tag{2.7}$$

Since

$$\begin{aligned}\frac{(rx^{\Delta})^{\Delta}(t)}{(rx^{\Delta})^{\sigma}(t)}\omega(t) &= \frac{(rx^{\Delta})^{\Delta}(t)}{r(t)x^{\Delta}(t)}\frac{r(t)x^{\Delta}(t)}{(rx^{\Delta})^{\sigma}(t)}\omega(t) = \frac{\omega^2(t)}{a(t)}\frac{r(t)x^{\Delta}(t)}{r(t)x^{\Delta}(t) + \mu(t)(rx^{\Delta})^{\Delta}(t)} \\ &= \frac{\omega^2(t)}{a(t)}\frac{1}{1 + \mu(t)(\omega(t)/a(t))} = \frac{\omega^2(t)}{a(t) + \mu(t)\omega(t)},\end{aligned}$$

we have that

$$\omega^{\Delta}(t) + \frac{x(\tau(t))}{(rx^{\Delta})^{\sigma}(t)}p(t) + \frac{\omega^2(t)}{a(t) + \mu(t)\omega(t)} = 0 \tag{2.8}$$

holds for $t \in [t_0, \infty)_{\mathbb{T}}$. From Lemma 2.2, we see that

$$\begin{aligned}\frac{x(\tau(t))}{(rx^{\Delta})^{\sigma}(t)} &= \frac{x(\tau(t))}{r(\tau(t))x^{\Delta}(\tau(t))}\frac{r(\tau(t))x^{\Delta}(\tau(t))}{(rx^{\Delta})^{\sigma}(t)} \\ &\geq \left(\frac{1}{\int_{t_1}^{\tau(t)}(\Delta s/a(s))}\int_{t_2}^{\tau(t)}\frac{\int_{t_1}^{s}(\Delta u/a(u))}{r(s)}\Delta s\right)\frac{\int_{t_1}^{\tau(t)}(\Delta s/a(s))}{\int_{t_1}^{\sigma(t)}(\Delta s/a(s))} = \frac{\int_{t_2}^{\tau(t)}\frac{\int_{t_1}^{s}(\Delta u/a(u))}{r(s)}\Delta s}{\int_{t_1}^{\sigma(t)}(\Delta s/a(s))}.\end{aligned}$$

Hence by (2.8), we obtain that (2.6) holds. Also, using Lemma 2.2,

$$\frac{(rx^{\Delta})^{\Delta}(t)}{(rx^{\Delta})^{\sigma}(t)} = \frac{(rx^{\Delta})^{\Delta}(t)}{r(t)x^{\Delta}(t)}\frac{r(t)x^{\Delta}(t)}{(rx^{\Delta})^{\sigma}(t)} \geq \frac{\omega(t)}{a(t)}\frac{\int_{t_1}^{t}(\Delta s/a(s))}{\int_{t_1}^{\sigma(t)}(\Delta s/a(s))}.$$

Thus, we have by (2.7) that (2.4) holds. Furthermore, since $a(rx^{\Delta})^{\Delta}$ is decreasing, we have

$$\frac{(rx^{\Delta})^{\Delta}(t)}{(rx^{\Delta})^{\sigma}(t)} = \frac{a(t)(rx^{\Delta})^{\Delta}(t)}{a(t)(rx^{\Delta})^{\sigma}(t)} \geq \frac{\omega^{\sigma}(t)}{a(t)},$$

and so (2.5) follows from (2.7). The proof is complete. □

Lemma 2.6. *Assume that x is a solution of* (1.12) *which satisfies case* (1) *of Lemma* 2.1. *Define the Riccati substitution ω as in* (2.3). *Then $\omega(t)\int_{t_2}^{t}(\Delta s/a(s)) \leq 1$ for $t \in [t_2, \infty)_{\mathbb{T}}$ and* $\lim_{t\to\infty}\omega(t) = 0$.

Proof. From (2.5), we have

$$\omega^{\Delta}(t) \leq -\frac{\omega^{\sigma}(t)\omega(t)}{a(t)}$$

for $t \in [t_2, \infty)_{\mathbb{T}}$, and so

$$\left(-\frac{1}{\omega}\right)^{\Delta}(t) = \frac{\omega^{\Delta}(t)}{\omega(t)\omega^{\sigma}(t)} \leq -\frac{1}{a(t)}$$

for $t \in [t_2, \infty)_{\mathbb{T}}$. Thus

$$\int_{t_2}^{t} \frac{\omega^{\Delta}(s)}{\omega(s)\omega^{\sigma}(s)}\Delta s \leq -\int_{t_2}^{t} \frac{\Delta s}{a(s)},$$

i.e.,

$$-\frac{1}{\omega(t)} + \frac{1}{\omega(t_2)} \leq -\int_{t_2}^{t} \frac{\Delta s}{a(s)},$$

which implies that $\omega(t)\int_{t_2}^{t}(\Delta s/a(s)) \leq 1$ for $t \in [t_2, \infty)_{\mathbb{T}}$ and $\lim_{t\to\infty}\omega(t) = 0$ due to condition (2.1). The proof is complete. □

Lemma 2.7. *Let x and ω be as in Lemma 2.5. Set*

$$p_* := \liminf_{t\to\infty} \int_{t_0}^{t} \frac{\Delta s}{a(s)} \int_{t}^{\infty} \frac{\int_{t_2}^{\tau(s)} \frac{\int_{t_1}^{v}(\Delta u/a(u))}{r(v)}\Delta v}{\int_{t_1}^{\sigma(s)}(\Delta u/a(u))} p(s)\Delta s, \tag{2.9}$$

$$q_* := \liminf_{t\to\infty} \frac{\int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)}(\Delta u/a(u))\right)^2 \left(\left(\int_{t_2}^{\tau(s)} \frac{\int_{t_1}^{u}(\Delta v/a(v))}{r(u)}\Delta u\right) / \left(\int_{t_1}^{\sigma(s)}(\Delta v/a(v))\right)\right) p(s)\Delta s}{\int_{t_0}^{t}(\Delta s/a(s))}, \tag{2.10}$$

$$r_* := \liminf_{t\to\infty} \omega(t)\int_{t_0}^{t} \frac{\Delta s}{a(s)}, \quad R_* := \limsup_{t\to\infty} \omega(t)\int_{t_0}^{t} \frac{\Delta s}{a(s)}, \tag{2.11}$$

$$l_* := \liminf_{t\to\infty} \frac{\left(\int_{t_0}^{\sigma(t)}(\Delta u/a(u))\right)^2 \left(\int_{t_1}^{t}(\Delta u/a(u))\right) / \left(\int_{t_1}^{\sigma(t)}(\Delta u/a(u))\right)}{\left(\int_{t_0}^{t}(\Delta s/a(s))\right)^2} \tag{2.12}$$

$$l^* := \limsup_{t\to\infty} \frac{\int_{t_0}^{\sigma(t)}(\Delta u/a(u))}{\int_{t_0}^{t}(\Delta s/a(s))}.$$

Then $0 \leq r_* \leq R_* \leq 1$ due to Lemma 2.6, $1 \leq l_* \leq l^* \leq \infty$, and

$$p_* \leq r_* - r_*^2, \quad q_* \leq \min\{1 - R_*, R_* l^* - r_*^2 l_*\}. \tag{2.13}$$

Proof. Multiplying (2.6) by $(\int_{t_0}^{\sigma(t)}(\Delta s/a(s)))^2$ and integrating the resulting inequality from t_3 ($t_3 \in [t_2, \infty)_{\mathbb{T}}$, $\tau(t) \in (t_2, \infty)_{\mathbb{T}}$ for $t \in [t_3, \infty)_{\mathbb{T}}$) to t, we see that

$$\begin{aligned}
&\int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)} \frac{\Delta u}{a(u)}\right)^2 \omega^{\Delta}(s)\Delta s + \int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)} \frac{\Delta u}{a(u)}\right)^2 \frac{\int_{t_2}^{\tau(s)} \frac{\int_{t_1}^{u}(\Delta v/a(v))}{r(u)}\Delta u}{\int_{t_1}^{\sigma(s)}(\Delta v/a(v))} p(s)\Delta s \\
&\quad + \int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)} \frac{\Delta u}{a(u)}\right)^2 \frac{\omega^2(s)}{a(s) + \mu(s)\omega(s)}\Delta s \leq 0.
\end{aligned} \tag{2.14}$$

An integration by parts in (2.14) yields

$$\begin{aligned}
&\left(\int_{t_0}^{t}\frac{\Delta s}{a(s)}\right)^2\omega(t)-\left(\int_{t_0}^{t_3}\frac{\Delta s}{a(s)}\right)^2\omega(t_3)-\int_{t_3}^{t}\left(\left(\int_{t_0}^{s}\frac{\Delta u}{a(u)}\right)^2\right)^{\Delta}\omega(s)\Delta s\\
&\quad+\int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}\right)^2\frac{\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{u}(\Delta v/a(v))}{r(u)}\Delta u}{\int_{t_1}^{\sigma(s)}(\Delta v/a(v))}p(s)\Delta s\\
&\quad+\int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}\right)^2\frac{\omega^2(s)}{a(s)+\mu(s)\omega(s)}\Delta s\le 0.
\end{aligned}$$

Since

$$\left(\left(\int_{t_0}^{s}\frac{\Delta u}{a(u)}\right)^2\right)^{\Delta}=\frac{1}{a(s)}\left[\int_{t_0}^{s}\frac{\Delta u}{a(u)}+\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}\right]=\frac{1}{a(s)}\left[2\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}-\frac{\mu(s)}{a(s)}\right],$$

we get after rearranging

$$\begin{aligned}
\left(\int_{t_0}^{t}\frac{\Delta s}{a(s)}\right)^2\omega(t)\le&\left(\int_{t_0}^{t_3}\frac{\Delta s}{a(s)}\right)^2\omega(t_3)-\int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}\right)^2\frac{\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{u}(\Delta v/a(v))}{r(u)}\Delta u}{\int_{t_1}^{\sigma(s)}(\Delta v/a(v))}p(s)\Delta s\\
&+\int_{t_3}^{t}H(s,\omega(s))\Delta s,
\end{aligned}\tag{2.15}$$

where

$$H(s,\omega(s)):=\frac{1}{a(s)}\left[2\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}-\frac{\mu(s)}{a(s)}\right]\omega(s)-\left(\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}\right)^2\frac{\omega^2(s)}{a(s)+\mu(s)\omega(s)}.$$

Now define

$$g(s,y):=\frac{1}{a(s)}\left[2\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}-\frac{\mu(s)}{a(s)}\right]y-\left(\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}\right)^2\frac{y^2}{a(s)+\mu(s)y}.$$

Then we obtain

$$\begin{aligned}
g(s,y)&=\frac{(a(s)+\mu(s)y)\left[2\int_{t_0}^{\sigma(s)}(\Delta u/a(u))-(\mu(s)/a(s))\right]y-a(s)\left(\int_{t_0}^{\sigma(s)}(\Delta u/a(u))\right)^2y^2}{a(s)(a(s)+\mu(s)y)}\\
&=\frac{\left[2\int_{t_0}^{\sigma(s)}(\Delta u/a(u))-(\mu(s)/a(s))\right]y-\left(\int_{t_0}^{s}(\Delta u/a(u))\right)^2y^2}{a(s)+\mu(s)y}.
\end{aligned}$$

We note that if $\mu(s)=0$, then we get $g(s,y)\le 1/a(s)$ (with respect to y). Furthermore in the case where $\mu(s)>0$, after some calculations, one can verify that for fixed $s>t_0$, the maximum of $g(s,y)$ for $y\ge 0$ occurs at $y_0=1/\int_{t_0}^{s}(\Delta u/a(u))$. Hence we obtain $g(s,y)\le 1/a(s)$ for $y\ge 0$. Thus, we see that $H(s,\omega(s))\le 1/a(s)$, and so

$$\int_{t_3}^{t}H(s,\omega(s))\Delta s\le\int_{t_3}^{t}\frac{\Delta s}{a(s)}.$$

Substituting the last inequality into (2.15) and dividing by $\int_{t_0}^{t}(\Delta s/a(s))$, we have

$$\begin{aligned}\omega(t)\int_{t_0}^{t}\frac{\Delta s}{a(s)} \leq & \frac{\left(\int_{t_0}^{t_3}(\Delta s/a(s))\right)^2\omega(t_3)}{\int_{t_0}^{t}(\Delta s/a(s))} \\ & - \frac{\int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)}(\Delta u/a(u))\right)^2\left(\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{u}(\Delta v/a(v))}{r(u)}\Delta u\right)/\left(\int_{t_1}^{\sigma(s)}(\Delta v/a(v))\right)p(s)\Delta s}{\int_{t_0}^{t}(\Delta s/a(s))} \\ & + \frac{\int_{t_3}^{t}(\Delta s/a(s))}{\int_{t_0}^{t}(\Delta s/a(s))}.\end{aligned} \tag{2.16}$$

If we take the lim sup of both sides of (2.16), we get

$$R_* \leq 1 - q_*.$$

Integrating (2.5) from t to ∞, we have by Lemma 2.6 that

$$\omega(t) \geq \int_{t}^{\infty}\frac{\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{v}(\Delta u/a(u))}{r(v)}\Delta v}{\int_{t_1}^{\sigma(s)}(\Delta u/a(u))}p(s)\Delta s + \int_{t}^{\infty}\frac{\omega^{\sigma}(s)\omega(s)}{a(s)}\Delta s. \tag{2.17}$$

Thus, from (2.17), we obtain

$$\begin{aligned}\omega(t)\int_{t_0}^{t}\frac{\Delta s}{a(s)} &\geq \int_{t_0}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\frac{\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{v}(\Delta u/a(u))}{r(v)}\Delta v}{\int_{t_1}^{\sigma(s)}(\Delta u/a(u))}p(s)\Delta s + \int_{t_0}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\frac{\omega^{\sigma}(s)\omega(s)}{a(s)}\Delta s \\ &= \int_{t_0}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\frac{\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{v}(\Delta u/a(u))}{r(v)}\Delta v}{\int_{t_1}^{\sigma(s)}(\Delta u/a(u))}p(s)\Delta s \\ &\quad + \int_{t_0}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\frac{\omega(s)\int_{t_0}^{s}(\Delta u/a(u))\omega^{\sigma}(s)\int_{t_0}^{\sigma(s)}(\Delta u/a(u))}{a(s)\int_{t_0}^{s}(\Delta u/a(u))\int_{t_0}^{\sigma(s)}(\Delta u/a(u))}\Delta s.\end{aligned}$$

Hence

$$\begin{aligned}\omega(t)\int_{t_0}^{t}\frac{\Delta s}{a(s)} \geq & \int_{t_0}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\frac{\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{v}(\Delta u/a(u))}{r(v)}\Delta v}{\int_{t_1}^{\sigma(s)}(\Delta u/a(u))}p(s)\Delta s \\ & + \int_{t_0}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\omega(s)\int_{t_0}^{s}\frac{\Delta u}{a(u)}\,\omega^{\sigma}(s)\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}A(s)\Delta s,\end{aligned} \tag{2.18}$$

where $A(s) := (-1/\int_{t_0}^{s}(\Delta u/a(u)))^{\Delta}$. Now for any $\varepsilon > 0$, there exists $t_4 \in (t_3, \infty)_{\mathbb{T}}$ such that

$$\omega(t)\int_{t_0}^{t}\frac{\Delta s}{a(s)} \geq r_* - \varepsilon$$

for all $t \in [t_4, \infty)_{\mathbb{T}}$. Hence from (2.18), we get

$$\omega(t)\int_{t_0}^{t}\frac{\Delta s}{a(s)} \geq \int_{t_0}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\frac{\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{v}(\Delta u/a(u))}{r(v)}\Delta v}{\int_{t_1}^{\sigma(s)}(\Delta u/a(u))}p(s)\Delta s + (r_* - \varepsilon)^2. \tag{2.19}$$

Therefore, taking the lim inf of both sides of (2.19) gives

$$r_* \geq p_* + (r_* - \varepsilon)^2.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$r_* \geq p_* + r_*^2.$$

Finally, we prove that

$$q_* \leq R_* l^* - r_*^2 l_*.$$

If $\varepsilon > 0$ is given arbitrarily, then there exists $t_3 \in (t_2, \infty)_{\mathbb{T}}$ such that

$$r_* - \varepsilon \leq \omega(t)\int_{t_0}^{t}\frac{\Delta s}{a(s)} \leq R_* + \varepsilon \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}},$$

$$\frac{\int_{t_0}^{\sigma(t)}(\Delta u/a(u))}{\int_{t_0}^{t}(\Delta s/a(s))} \leq l^* + \varepsilon \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}}$$

and

$$\frac{\left(\int_{t_0}^{\sigma(t)}(\Delta u/a(u))\right)^2 \frac{\int_{t_1}^{t}(\Delta u/a(u))}{\int_{t_1}^{\sigma(t)}(\Delta u/a(u))}}{\left(\int_{t_0}^{t}(\Delta s/a(s))\right)^2} \geq l_* - \varepsilon \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}}.$$

Using (2.4) and a similar proof of (2.15), we have

$$\begin{aligned}\left(\int_{t_0}^{t}\frac{\Delta s}{a(s)}\right)^2\omega(t) \leq &\left(\int_{t_0}^{t_3}\frac{\Delta s}{a(s)}\right)^2\omega(t_3) + \int_{t_3}^{t}\frac{\left[\int_{t_0}^{s}(\Delta u/a(u)) + \int_{t_0}^{\sigma(s)}(\Delta u/a(u))\right]\omega(s)}{a(s)}\Delta s \\ &- \int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}\right)^2\left(\int_{t_0}^{\tau(s)}\frac{\int_{t_1}^{u}(\Delta v/a(v))}{r(u)}\Delta u\right)\Big/\left(\int_{t_1}^{\sigma(s)}(\Delta v/a(v))\right)p(s)\Delta s \\ &- \int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)}\frac{\Delta u}{a(u)}\right)^2\frac{\omega^2(s)}{a(s)}\frac{\int_{t_1}^{s}(\Delta u/a(u))}{\int_{t_1}^{\sigma(s)}(\Delta u/a(u))}\Delta s.\end{aligned}$$

That is,

$$\omega(t)\int_{t_0}^{t}\frac{\Delta s}{a(s)} \leq \frac{\left(\int_{t_0}^{t_3}(\Delta s/a(s))\right)^2\omega(t_3)}{\int_{t_0}^{t}(\Delta s/a(s))} + \frac{\int_{t_3}^{t}(1/a(s))\left[\int_{t_0}^{s}(\Delta u/a(u)) + \int_{t_0}^{\sigma(s)}(\Delta u/a(u))\right]\omega(s)\Delta s}{\int_{t_0}^{t}(\Delta s/a(s))}$$
$$- \frac{\int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)}(\Delta u/a(u))\right)^2\left(\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{u}(\Delta v/a(v))}{r(u)}\Delta u\right)/\left(\int_{t_1}^{\sigma(s)}(\Delta v/a(v))\right)p(s)\Delta s}{\int_{t_0}^{t}(\Delta s/a(s))}$$
$$- \frac{\int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)}(\Delta u/a(u))\right)^2(\omega^2(s)/a(s))\left(\int_{t_1}^{s}(\Delta u/a(u))\right)/\left(\int_{t_1}^{\sigma(s)}(\Delta u/a(u))\right)\Delta s}{\int_{t_0}^{t}(\Delta s/a(s))}.$$

Then we obtain

$$\omega(t)\int_{t_0}^{t}\frac{\Delta s}{a(s)} \leq \frac{\left(\int_{t_0}^{t_3}(\Delta s/a(s))\right)^2\omega(t_3)}{\int_{t_0}^{t}\frac{\Delta s}{a(s)}} + \frac{\int_{t_3}^{t}\frac{(1/a(s))\left[\int_{t_0}^{s}(\Delta u/a(u))+\int_{t_0}^{\sigma(s)}(\Delta u/a(u))\right]\omega(s)\int_{t_0}^{s}(\Delta u/a(u))}{\int_{t_0}^{s}(\Delta u/a(u))}\Delta s}{\int_{t_0}^{t}(\Delta s/a(s))}$$
$$- \frac{\int_{t_3}^{t}\left(\int_{t_0}^{\sigma(s)}(\Delta u/a(u))\right)^2\left(\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{u}(\Delta v/a(v))}{r(u)}\Delta u\right)/\left(\int_{t_1}^{\sigma(s)}(\Delta v/a(v))\right)p(s)\Delta s}{\int_{t_0}^{t}(\Delta s/a(s))}$$
$$- \frac{\int_{t_3}^{t}\frac{\left(\int_{t_0}^{\sigma(s)}(\Delta u/a(u))\right)^2\left(\left(\int_{t_1}^{s}(\Delta u/a(u))\right)/\left(\int_{t_1}^{\sigma(s)}(\Delta u/a(u))\right)\right)}{\left(\int_{t_0}^{s}(\Delta u/a(u))\right)^2}(\omega^2(s)/a(s))\left(\int_{t_0}^{s}(\Delta u/a(u))\right)^2\Delta s}{\int_{t_0}^{t}(\Delta s/a(s))}$$
$$\leq \frac{\left(\int_{t_0}^{t_3}(\Delta s/a(s))\right)^2\omega(t_3)}{\int_{t_0}^{t}(\Delta s/a(s))} + (R_* + \varepsilon)(1 + l^* + \varepsilon)\frac{\int_{t_3}^{t}(\Delta s/a(s))}{\int_{t_0}^{t}(\Delta s/a(s))}$$
$$- (r_* - \varepsilon)^2(l_* - \varepsilon)\frac{\int_{t_3}^{t}(\Delta s/a(s))}{\int_{t_0}^{t}(\Delta s/a(s))} - q_*.$$

Taking the lim sup of both sides of the last inequality, we obtain

$$R_* \leq R_*(1 + l^*) - r_*^2 l_* - q_*$$

due to $\varepsilon > 0$ is arbitrary, which yields the desired result. The proof is complete. $\square$

Now we may establish some oscillation criteria based on the previous lemmas.

THEOREM 2.8. *Assume that* (2.1) *and* (2.2) *hold and let x be a solution of* (1.12). *If*

$$p_* = \liminf_{t\to\infty}\int_{t_0}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\frac{\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{v}(\Delta u/a(u))}{r(v)}\Delta v}{\int_{t_1}^{\sigma(s)}(\Delta u/a(u))}p(s)\Delta s > \frac{1}{4}, \tag{2.20}$$

then x is oscillatory or satisfies $\lim_{t\to\infty}x(t) = 0$.

Proof. Suppose that x is a nonoscillatory solution of (1.12). Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Then if case (1) of Lemma 2.1 holds, let ω be as defined in (2.3). From Lemma 2.7, we see that

$$p_* \leq r_* - r_*^2 \leq \frac{1}{4},$$

which contradicts (2.20). Now if case (2) of Lemma 2.1 holds, then by Lemma 2.3, $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

THEOREM 2.9. *Assume that* (2.1) *and* (2.2) *hold and let x be a solution of* (1.12). *If*

$$q_* = \liminf_{t\to\infty} \frac{\int_{t_3}^{t} \left(\int_{t_0}^{\sigma(s)} (\Delta u/a(u))\right)^2 \left(\int_{t_2}^{\tau(s)} \frac{\int_{t_1}^{u} (\Delta v/a(v))}{r(u)} \Delta u\right) \Big/ \left(\int_{t_1}^{\sigma(s)} (\Delta v/a(v))\right) p(s)\Delta s}{\int_{t_0}^{t} (\Delta s/a(s))} > \frac{l^*}{1+l^*}, \quad (2.21)$$

then x is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that x is a nonoscillatory solution of (1.12). Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Then if case (1) of Lemma 2.1 holds, let ω be as defined in (2.3). From Lemma 2.7, we get that

$$q_* \leq \min\{1 - R_*, R_* l^* - r_*^2 l_*\} \leq \min\{1 - R_*, R_* l^*\},$$

which yields

$$q_* \leq \frac{l^*}{1+l^*},$$

which contradicts (2.21). Now if case (2) of Lemma 2.1 holds, then by Lemma 2.3, $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

THEOREM 2.10. *Assume that* (2.1) *and* (2.2) *hold and let x be a solution of* (1.12). *If* $0 \leq p_* \leq 1/4$ *and*

$$\begin{aligned} q_* = \liminf_{t\to\infty} &\frac{\int_{t_3}^{t} \left(\int_{t_0}^{\sigma(s)} (\Delta u/a(u))\right)^2 \left(\int_{t_2}^{\tau(s)} \frac{\int_{t_1}^{u} (\Delta v/a(v))}{r(u)} \Delta u\right) \Big/ \left(\int_{t_1}^{\sigma(s)} (\Delta v/a(v))\right) p(s)\Delta s}{\int_{t_0}^{t} (\Delta s/a(s))} \\ &> \frac{l^* - \left((1/2) - p_* - (1/2)\sqrt{1-4p_*}\right) l_*}{1+l^*}, \end{aligned} \quad (2.22)$$

then x is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that x is a nonoscillatory solution of (1.12). Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Then if case (1) of Lemma 2.1 holds, let ω be as defined in (2.3). From Lemma 2.7, we use the fact that $b := p_* \leq r_* - r_*^2$

to get that

$$r_* \geq r_0 := \frac{1}{2} - \frac{\sqrt{1-4b}}{2}.$$

Using (2.13), we have

$$q_* \leq \min\{1 - R_*, R_* l^* - r_*^2 l_*\} \leq \min\{1 - R_*, R_* l^* - r_0^2 l_*\}$$

for $r_0 \leq R_* \leq 1$, which implies that

$$q_* \leq \frac{l^* - \big((1/2) - p_* - (1/2)\sqrt{1-4p_*}\big) l_*}{1 + l^*},$$

after some calculations. This contradicts (2.22). Now if case (2) of Lemma 2.1 holds, then by Lemma 2.3, $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

3. Discussions

Remark 3.1. In this paper, we suggest some new Hille and Nehari type asymptotic criteria for a third-order delay dynamic equation (1.12), thereinto, Theorems 2.8, 2.9 and 2.10 include [15, Theorems 2, 3 and 4], respectively. Some examples may be given by consulting those in [15], and hence are omitted.

Remark 3.2. Regarding the Hille and Nehari type criteria for asymptotic properties of third-order dynamic equations on time scales, see [15,31,33]. However, as mentioned in Section 1, there exist some restrictions, e.g., (1.7) some restrictive assumptions on coefficients and any auxiliary functions. In this paper, we remove auxiliary function $h_2(t, t_0)$ as in [15] and extend its results to a more general delay dynamic equation (1.12); we delete assumption (1.7) as in [33]; also the main results do not need condition $a^{\Delta}(t) \geq 0$ and $r(t) = 1$ as in [31,32]. Furthermore, one can easily see that our results improve those by [9,12,13,14,20,21,25,26,34].

Remark 3.3. The question regarding the study of sufficient conditions which guarantee that all solutions of (1.12) oscillate remains open at the moment. It is well known (see [10]) that the solutions of third-order Euler differential equation

$$x'''(t) + \frac{\beta}{t^3} x(t) = 0$$

are oscillatory when $\beta > 2/(3\sqrt{3})$. How to extend this sharp criterion to third-order dynamic equations on time scales also remains open.

Acknowledgement

This research is supported by NNSF of P.R. China (Grant Nos 61034007, 51277116 and 50977054).

Notes

1. Email: agarwal@tamuk.edu.
2. Email: litongx2007@163.com.
3. Email: zchui@sdu.edu.cn.

References

[1] R.P. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, *Dynamic equations on time scales: A survey*, J. Comput. Appl. Math. 141 (2002), pp. 1–26.
[2] R.P. Agarwal, M. Bohner, and S.H. Saker, *Oscillation criteria for second order delay dynamic equations*, Can. Appl. Math. Q. 13 (2005), pp. 1–17.
[3] R.P. Agarwal, M. Bohner, S. Tang, T. Li, and C. Zhang, *Oscillation and asymptotic behavior of third-order nonlinear retarded dynamic equations*, Appl. Math. Comput. 219 (2012), pp. 3600–3609.
[4] R.P. Agarwal, D. O'Regan, and S.H. Saker, *Oscillation criteria for second-order nonlinear neutral delay dynamic equations*, J. Math. Anal. Appl. 300 (2004), pp. 203–217.
[5] E. Akın-Bohner, M. Bohner, and S.H. Saker, *Oscillation criteria for a certain class of second order Emden-Fowler dynamic equations*, Electron. Trans. Numer. Anal. 27 (2007), pp. 1–12.
[6] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
[7] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
[8] M. Bohner and S.H. Saker, *Oscillation of second order nonlinear dynamic equations on time scales*, Rocky Mountain J. Math. 34 (2004), pp. 1239–1254.
[9] E.M. Elabbasy and T.S. Hassan, *Oscillation of solutions for third order functional dynamic equations*, Electron. J. Diff. Equ. 131 (2010), pp. 1–14.
[10] L. Erbe, *Existence of oscillatory solutions and asymptotic behavior for a class of third order linear differential equations*, Pacific J. Math. 64 (1976), pp. 369–385.
[11] L. Erbe, T.S. Hassan, and A. Peterson, *Oscillation of third order nonlinear functional dynamic equations on time scales*, Differ. Equ. Dyn. Syst. 18 (2010), pp. 199–227.
[12] L. Erbe, T.S. Hassan, and A. Peterson, *Oscillation of third order functional dynamic equations with mixed arguments on time scales*, J. Appl. Math. Comput. 34 (2010), pp. 353–371.
[13] L. Erbe, A. Peterson, and S.H. Saker, *Asymptotic behavior of solutions of a third-order nonlinear dynamic equation on time scales*, J. Comput. Appl. Math. 181 (2005), pp. 92–102.
[14] L. Erbe, A. Peterson, and S.H. Saker, *Oscillation and asymptotic behavior of a third-order nonlinear dynamic equation*, Can. Appl. Math. Q. 14 (2006), pp. 129–147.
[15] L. Erbe, A. Peterson, and S.H. Saker, *Hille and Nehari type criteria for third-order dynamic equations*, J. Math. Anal. Appl. 329 (2007), pp. 112–131.
[16] L. Erbe, A. Peterson, and S.H. Saker, *Oscillation criteria for second-order nonlinear delay dynamic equations*, J. Math. Anal. Appl. 333 (2007), pp. 505–522.
[17] S.R. Grace, R.P. Agarwal, M. Bohner, and D. O'Regan, *Oscillation of second-order strongly superlinear and strongly sublinear dynamic equations*, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009), pp. 3463–3471.
[18] S.R. Grace, M. Bohner, and R.P. Agarwal, *On the oscillation of second-order half-linear dynamic equations*, J. Diff. Equ. Appl. 15 (2009), pp. 451–460.
[19] S.R. Grace, M. Bohner, and A. Liu, *On Kneser solutions of third-order delay dynamic equations*, Carpathian J. Math. 26 (2010), pp. 184–192.
[20] Z. Han, T. Li, S. Sun, and F. Cao, *Oscillation criteria for third order nonlinear delay dynamic equations on time scales*, Ann. Polon. Math. 99 (2010), pp. 143–156.
[21] T.S. Hassan, *Oscillation of third order nonlinear delay dynamic equations on time scales*, Math. Comput. Model. 49 (2009), pp. 1573–1586.
[22] S. Hilger, *Analysis on measure chains – A unified approach to continuous and discrete calculus*, Results Math. 18 (1990), pp. 18–56.
[23] I. Kubiaczyk and S.H. Saker, *Asymptotic properties of third order functional dynamic equations on time scales*, Ann. Polon. Math. 100 (2011), pp. 203–222.
[24] T. Li, R.P. Agarwal, and M. Bohner, *Some oscillation results for second-order neutral dynamic equations*, Hacet. J. Math. Stat. 41(5) (2012), pp. 715–721.

[25] T. Li, Z. Han, S. Sun, and Y. Zhao, *Oscillation results for third order nonlinear delay dynamic equations on time scales*, Bull. Malays. Math. Sci. Soc. 34 (2011), pp. 639–648.

[26] T. Li, Z. Han, Y. Sun, and Y. Zhao, *Asymptotic behavior of solutions for third-order half-linear delay dynamic equations on time scales*, J. Appl. Math. Comput. 36 (2011), pp. 333–346.

[27] P. Řehák, *How the constants in Hille-Nehari theorems depend on time scales*, Adv. Diff. Equ. (2006) (2006), pp. 1–15.

[28] Y. Şahiner, *Oscillation of second-order delay differential equations on time scales*, Nonlinear Anal. TMA 63 (2005), pp. 1073–1080.

[29] S.H. Saker, *Oscillation of nonlinear dynamic equations on time scales*, Appl. Math. Comput. 148 (2004), pp. 81–91.

[30] S.H. Saker, *Oscillation criteria of second-order half-linear dynamic equations on time scales*, J. Comput. Appl. Math. 177 (2005), pp. 375–387.

[31] S.H. Saker, *Oscillation of third-order functional dynamic equations on time scales*, Sci. China Math. 54 (2011), pp. 2597–2614.

[32] S.H. Saker, *On oscillation of a certain class of third-order nonlinear functional dynamic equations on time scales*, Bull. Math. Soc. Sci. Math. Roumanie Tome 54 (2011), pp. 365–389.

[33] Y. Wang and Z. Xu, *Asymptotic properties of solutions of certain third-order dynamic equations*, J. Comput. Appl. Math. 236 (2012), pp. 2354–2366.

[34] Z. Yu and Q. Wang, *Asymptotic behavior of solutions of third-order nonlinear dynamic equations on time scales*, J. Comput. Appl. Math. 225 (2009), pp. 531–540.

[35] C. Zhang, R.P. Agarwal, M. Bohner, and T. Li, *New oscillation results for second-order neutral delay dynamic equations*, Adv. Difference Equ. 2012(227) (2012), pp. 1–14.